

THE SCHWARZIAN DERIVATIVE AND POLYNOMIAL ITERATION

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ABSTRACT. We consider the Schwarzian derivative S_f of a complex polynomial f and its iterates. We show that the sequence S_{f^n}/d^{2n} converges to $-2(\partial G_f)^2$, for G_f the escape-rate function of f . As a quadratic differential, the Schwarzian derivative S_{f^n} determines a conformal metric on the plane. We study the ultralimit of these metric spaces.

1. INTRODUCTION

Recall that the Schwarzian derivative of a holomorphic function f on the complex plane is defined as

$$S_f(z) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2$$

It is well known that $S_f \equiv 0$ if and only if f is a Möbius transformation. We can view the Schwarzian derivative as a measure of the complexity of a nonconstant holomorphic function.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a complex polynomial with degree $d \geq 2$. In this article we examine the sequence of Schwarzian derivatives of the iterates f^n (f composed with itself n times) of f . Specifically, we look at the sequence

$$\left\{ \frac{S_{f^n}(z)}{d^{2n}} \right\}_{n \geq 1}$$

and view it as a sequence of meromorphic functions or quadratic differentials on the Riemann sphere. We are interested in understanding the limit as $n \rightarrow \infty$.

We begin with the simplest example.

Example 1. Let $f(z) = z^d$ with $d \geq 2$, then we get

$$\begin{aligned} S_{f^n}(z) &= \frac{2(d^n-1)(d^n-2)-3(d^n-1)^2}{2z^2} \\ &= \frac{1-d^{2n}}{2z^2} \end{aligned}$$

Since $d \geq 2$, the sequence of normalized Schwarzians converges,

$$\lim_{n \rightarrow \infty} \frac{S_{f^n}}{d^{2n}} = \lim_{n \rightarrow \infty} \frac{1-d^{2n}}{2d^{2n}z^2} = -\frac{1}{2z^2}$$

locally uniformly on $\mathbb{C} \setminus \{0\}$.

The normalized Schwarzians as quadratic differentials $\left\{ \frac{S_{f^n}}{d^{2n}} dz^2 \right\}$ converge to $-\frac{1}{2z^2} dz^2$. The associated conformal metric $ds = \frac{|dz|}{\sqrt{2z}}$ makes $\mathbb{C} \setminus \{0\}$ isometric to an infinite cylinder of radius $\frac{1}{\sqrt{2}}$. The cylinder's closed geodesics are the horizontal trajectories of the quadratic differential $-\frac{1}{2z^2} dz^2$.

Local convergence. Let G_f be the escape-rate function of f , which is defined as

$$G_f = \lim_{n \rightarrow \infty} \frac{\log^+ |f^n|}{d^n},$$

where $\log^+ |x| = \max(\log |x|, 0)$. Let $\text{Precrit}(f) = \cup_{n>0} \{c \in \mathbb{C} \mid (f^n)'(c) = 0\}$ be the union of the critical points of f and their backward orbits. Note that its closure $\overline{\text{Precrit}(f)}$ contains the Julia set $J(f)$ when f is not conjugate to z^d .

Theorem 1.1. *Let f be a polynomial with degree $d \geq 2$ and not conformally conjugate to z^d . Then the sequence of Schwarzian derivatives S_{f^n} satisfies*

$$\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} = -2 \left(\frac{\partial G_f(z)}{\partial z} \right)^2,$$

locally uniformly on $\mathbb{C} \setminus \overline{\text{Precrit}(f)}$.

Remark. The choice of normalization $\frac{1}{d^{2n}}$ allows us to focus on the basin of infinity. Other normalizations might detect interesting properties of $J(f)$. In Corollary 3.4, we show that $\left\{ \frac{S_{f^n}}{d^{2n}} dz^2 \right\}$ converge on the entire Fatou set, in the sense of $L_{loc}^{\frac{1}{2}}$ convergence.

Sometimes, people are also interested in the nonlinearity of a nonconstant holomorphic function on \mathbb{C} . Similar with Theorem 1.1, we have $\lim_{n \rightarrow \infty} \frac{N_{f^n} dz}{d^n} = \partial G_f$ locally uniformly on $\mathbb{C} \setminus \overline{\text{Precrit}(f)}$, where $N_f = f'/f''$; see Theorem 3.3.

Metric space convergence. Let f be a polynomial with degree $d \geq 2$. Each S_{f^n} determines a conformal geodesic metric d_n on the complement of the critical points of f^n , given by $ds = \sqrt{|-\frac{S_{f^n}}{d^{2n}} dz^2|}$. From this sequence of geodesic spaces, we obtain an ultralimit $(X_\omega, d_\omega, a_\omega)$; see Chapter I §5 [BH] for more details about the ultralimit. The limit space is a complete geodesic space.

The escape-rate function G_f also determines a conformal metric on the basin of infinity $X_o = \{z \in \mathbb{C} \mid f^n(z) \rightarrow \infty\}$ of f , given by $ds = \sqrt{2} |\partial G_f|$. Given a choice of base point $a \in X_o \setminus \overline{\text{Precrit}(f)}$, we denote this pointed metric space by (X_o, d_o, a) ; compare [DM] or [DP] where this metric already appeared.

When the Julia set is not connected, the ultralimit X_ω can be described as a “hairy” version of X_o .

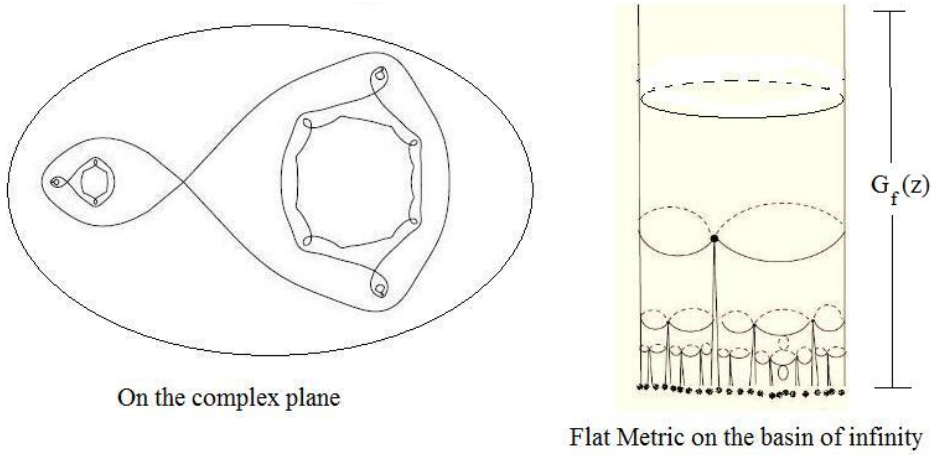


FIGURE 1.1. Level set structure of G_f for a cubic polynomial and flat metric structure on the basin of infinity

Theorem 1.2. *Let f be a polynomial with degree $d \geq 2$ and disconnected Julia set. There exists a natural embedding from $X_o(f) \setminus \text{Precrit}(f)$ to X_ω which extends to the metric completion (\overline{X}_o, d_o, a) as an isometric embedding.*

The metric space X_ω is obtained by attaching a real ray to \overline{X}_o at each point in $\text{Precrit}(f)$, and attaching some non trivial space (containing infinitely many real rays) to each connected component of $\overline{X}_o \setminus X_o$.

Remark. For $J(f)$ disconnected, we use the tree structure on the basin of infinity to show the embedding is an isometry. For $J(f)$ connected, we also have a locally isometric embedding of X_o to X_ω ; this follows easily from the argument in the proof of the above theorem. But we do not expect the embedding to be a global isometry. The reason we use the ultralimit to study the limiting space is that the spaces are not compact and the metrics d_n are not uniformly proper, so more classical notions of convergence like Gromov-Hausdorff convergence won't work. We are not quite sure whether the ultralimit X_ω depends on the ultrafilter ω or not. But from the above theorem, the only things that might depend on the ultrafilter are the spaces attached to $\overline{X}_o \setminus X_o$.

Conjugacy classes. The study of $\{S_{f^n}\}$ has grown out of an attempt to better understand the moduli space M_d of polynomials (the space of conformal conjugacy classes). The geometric structure $(f, X_o, |\partial G_f|)$ has been studied in [DP] and used to classify topological conjugacy classes. We can also use the Schwarzian derivative to classify polynomials with the same degree. We define an equivalence relation on the set of polynomials with degree $d \geq 2$ as: $f \sim g$ if $S_f dz^2 = A^*(S_g dz^2)$, for $A(z) = az+b$ some affine transformation. From this definition, polynomials f and g are equivalent to each other if and only if $f = B \circ g \circ A$ with A and B affine transformations, and if and only if f and g have the same critical set (counted with multiplicities) up to

some affine transformation, i.e. $A(\text{Crit}(f)) = \text{Crit}(g)$ for some affine transformation A . See Lemma 2.2 for details. Note that affine conjugate polynomials are equivalent in this sense.

Theorem 1.3. *Let f and g be polynomials with the same degree $d \geq 2$. Then the following are equivalent:*

- $f^n \sim g^n$ for infinitely many $n \in \mathbb{N}^*$.
- $f^n \sim g^n$ for all $n \in \mathbb{N}^*$.
- f and g have the same Julia set up to some affine transformation ($A(J(g)) = J(f)$ with A affine transformation).

Polynomials f and g which satisfy the above conditions are called strongly equivalent. Each such strong equivalence class consists of finitely many affine conjugacy classes (no more than the order of the symmetry group of the Julia set).

Notes: the above theorem relies on the classification of polynomials with the same Julia set, and the proof uses the main result of [Be2] (see §2.2 for other references and historical context).

I would like to thank David Dumas, Kevin Pilgrim, Stefan Wenger, Curt McMullen and especially Laura DeMarco for lots of helpful comments and suggestions.

2. BASIC PROPERTIES OF SCHWARZIAN DERIVATIVES

In this section, we give useful formulas for S_{f^n} and also basic definitions that we are going to use later in this article.

2.1. Basic formula for the Schwarzian derivative of f^n . In order to find the limit of $\frac{S_{f^n}}{d^{2n}}$ for a general polynomial with degree $d \geq 2$, we need to rewrite S_{f^n} in terms of S_f and then evaluate the limit. To do this, we use the formula for the Schwarzian derivative of the composition of two functions f and g . An easy calculation shows that (cocycle property)

$$(2.1) \quad S_{f \circ g}(z) = S_f(g(z))(g'(z))^2 + S_g(z)$$

From this relation we derive the following important formula:

$$\begin{aligned}
 S_{f^n}(z) &= S_f(f^{n-1}(z))((f^{n-1}(z))')^2 + S_{f^{n-1}}(z) \\
 (2.2) \quad &= S_f(f^{n-1}(z))((f^{n-1}(z))')^2 + S_f(f^{n-2}(z))((f^{n-2}(z))')^2 + S_{f^{n-2}}(z) \\
 &= \sum_{i=1}^{n-1} S_f \circ f^i(z)((f^i(z))')^2 + S_f(z)
 \end{aligned}$$

Proposition 2.1. *Let f be a polynomial with degree $d \geq 2$. For any point $z \in \mathbb{C} \setminus \text{Precrit}(f)$ and any sequence $\{n_i\}_{i=1}^\infty$ of \mathbb{N} with $n_i \rightarrow \infty$ as $i \rightarrow \infty$, the sequences $\{\frac{S_{f^{n_i}}(z)}{d^{2n_i}}\}$ and $\{\frac{S_{f^{n_i-1}}(f(z))}{d^{2(n_i-1)}}\}$ either both converge, diverge to infinity or diverge.*

Moreover, if both of them converge, then

$$d^2 \lim_{i \rightarrow \infty} \frac{S_{f^{n_i}}(z)}{d^{2n_i}} = (f'(z))^2 \lim_{i \rightarrow \infty} \frac{S_{f^{n_i-1}}(f(z))}{d^{2(n_i-1)}}.$$

Proof. From (2.1), we have:

$$\begin{aligned} \frac{S_{f^{n_i}}(z)}{d^{2n_i}} &= \frac{S_{f^{n_i-1}}(f(z))(f'(z))^2 + S_f(z)}{d^{2n_i}} \\ &= \frac{S_{f^{n_i-1}}(f(z))(f'(z))^2}{d^2 d^{2(n_i-1)}} + \frac{S_f(z)}{d^{2n_i}} \end{aligned}$$

By the assumption that z is not a critical point of f and $d \geq 2$, it is easy to see that this proposition is satisfied since $f'(z)$ is not equal to zero and $S_f(z)$ is finite. \square

2.2. The Schwarzian derivative as a quadratic differential. In this subsection, we are not only considering polynomials, but also rational maps with degree $d \geq 2$.

Meromorphic quadratic differentials. For any Riemann surface S , a meromorphic quadratic differential Q on S is a section of the second tensor power of the cotangent bundle. In local coordinate, $Q = Q_1(z)dz^2$, where $Q_1(z)$ is a meromorphic function. And under changing of coordinate $w = w(z)$,

$$Q = Q_2(w)(dw)^2 = Q_2(w(z))(w'(z))^2 dz^2$$

i.e., $Q_2(w(z))(w'(z))^2 = Q_1(z)$.

Consider a non constant holomorphic map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. This is a rational map with finite degree. Let's look at the Schwarzian derivative of this rational map f , and view it as quadratic differential, i.e.

$$S_f dz^2 \text{ instead of } S_f(z)$$

From the definition of the Schwarzian derivative, it is not hard to show $S_g \equiv 0$ if and only if g is a Möbius transformation; see [Du]. From the following identity

$$S_{f \circ g}(z) dz^2 = S_f(g(z))(dg)^2 + S_g(z) dz^2,$$

for any two Möbius transformations g_0, g_1 , we have

$$S_{g_1 \circ f \circ g_0} dz^2 = S_f \circ g_0 (dg_0)^2$$

So the Schwarzian derivative $S_f dz^2$ as a quadratic differential is well defined on \mathbb{P}^1 . More generally, for any non constant holomorphic map from some projective Riemann surface to another projective Riemann surface, there is an unique quadratic differential (named as Schwarzian derivative) associated to it; see [Du].

Recall that in the last part of the introduction, we defined an equivalence relation of polynomials: $f \sim g$ if the Schwarzian derivative of f is the same as the Schwarzian derivative of g up to some affine transformation. The Schwarzian derivative of a polynomial is determined by the locations and multiplicities of the critical points:

Lemma 2.2. *Let f and g be polynomials with degree $d \geq 2$. Then $f \sim g$ if and only if they have the same critical set (critical points are counted with multiplicity) up to some affine transformation ($A(\text{Crit}(f)) = \text{Crit}(g)$ with A some affine transformation).*

Proof. Assume $f \sim g$, then $S_f dz^2 = A^*(S_g dz^2)$, which means $f = B \circ g \circ A$ for some affine transformations A and B . Indeed, by the cocycle property, $A^*(S_g dz^2) = S_{g \circ A} dz^2$ and then $S_{f \circ (g \circ A)^{-1}} \equiv 0$ on some open subset of \mathbb{C} . So $B = f \circ (g \circ A)^{-1}$ is a Möbius transformation on some open subset of \mathbb{C} . By continuity, $f = B \circ g \circ A$ in \mathbb{C} , and so B is an affine transformation. This implies that A transforms the critical set of f to the critical set of g .

Conversely, assume that there is an affine transformation A that transforms the critical set of f to the critical set of g . Since $g \sim g \circ A$, it suffices to show that $f \sim g \circ A$. Because f and $g \circ A$ have the same critical set, so we can let the critical set be $\{c_i\}_{i=1}^{d-1}$. Then $f = ah(z) + b$ and $g \circ A = ch(z) + d$ with $a, c \neq 0$ and $h(z) = \int_0^z \prod_{i=1}^{d-1} (t - c_i) dt$. Which means $S_f = S_h = S_{g \circ A}$, i.e. $f \sim g \circ A \sim g$. \square

Proof of Theorem 1.3. Let \simeq be the notion of strong equivalence. Assume that the polynomial f with degree $d \geq 2$ is not conjugate to z^d , and there is a subsequence $\{n_i\}_{i=1}^\infty \subset \mathbb{N}^*$ such that $f^{n_i} \sim g^{n_i}$. By Lemma 2.2, there are affine transformations $\{A_i = a_i z + b_i\}$ such that $A_i(\text{Crit}(f^{n_i})) = \text{Crit}(g^{n_i})$. Since f is not conjugate to z^d , so g is not conjugate to z^d . Indeed, for $n \geq 2$, $\text{Crit}(f^n)$ has at least two distinct points, however, $\text{Crit}(z^{d^n})$ has only one point. Set $M_1 = \text{Diam}(\text{Crit}(f^2)) > 0$, $M'_1 = \text{Diam}(\text{Crit}(g^2)) > 0$, $M_2 = \text{Diam}(\text{Precrit}(f))$ and $M'_2 = \text{Diam}(\text{Precrit}(g))$. Because $f(\mathbb{C} \setminus D(0, R)) \subset \mathbb{C} \setminus D(0, R)$ for R sufficiently large, so $\text{Precrit}(f)$ is bounded and then $M_2 < \infty$. Similarly, $M'_2 < \infty$. Moreover, since $f^{n+m} = f^n \circ f^m$, then $\text{Crit}(f^m) \subset \text{Crit}(f^{n+m})$. So for the diameters $\text{Diam}(\text{Crit}(f^{n_i}))$ and $\text{Diam}(\text{Crit}(g^{n_i}))$ of the critical sets, we have

$$0 < M_1 \leq \text{Diam}(\text{Crit}(f^{n_i})) \leq M_2 < \infty, \quad 0 < M'_1 \leq \text{Diam}(\text{Crit}(g^{n_i})) \leq M'_2 < \infty,$$

for any $n_i \geq 2$. Consequently,

$$0 < M_3 \leq |a_i| \leq M_4 \leq \infty, \quad |b_i| \leq M_5 < \infty$$

So after passing to a subsequence and without loss of generality, we can assume $A_i \rightarrow A$ as $i \rightarrow \infty$, where A is an affine transformation. Then it is easy to know $A(J(f)) = J(g)$. Indeed, for any $c \in \text{Crit}(f^j) \subset \text{Crit}(f^i)$, $A_i(c) \in \text{Crit}(g^i) \subset \text{Precrit}(g)$ with $j \leq i$. It indicates that $A(c) \in \overline{\text{Precrit}(g)}$ and then $A(\overline{\text{Precrit}(f)}) \subset \overline{\text{Precrit}(g)}$. Similarly, by taking A_i^{-1} instead of A_i , we get $A^{-1}(\overline{\text{Precrit}(g)}) \subset \overline{\text{Precrit}(f)}$. Because f is not

conjugate to z^d , the set of accumulating points of $\overline{\text{Precrit}(f)}$ (respt. $\overline{\text{Precrit}(g)}$) is $J(f)$ (respt. $J(g)$), which means that $A(J(f)) = J(g)$.

If f is conjugate to z^d , then by the above argument, g should also be conjugate to z^d . So there is an affine map A such that $A(J(f)) = J(g)$.

Conversely, assume f and g have the same degree $d \geq 2$ and $A(J(f)) = J(g)$ for some affine transformation A . Since $g_1 = A^{-1} \circ g \circ A \simeq g$ and $A(J(g_1)) = J(g) = A(J(f))$, so it is enough to prove that $g_1 \simeq f$ with the condition that they have the same Julia set. First, if the Julia set is a circle, then both of them are conjugate to z^d . Indeed, we can assume $J(f)$ is the closed unit disk. Let φ be the Boettcher function of f , such that $\varphi \circ f \circ \varphi^{-1} = z^d$ on the basin of infinity. Since the basin of infinity is the complement of the unit disk and φ is a conformal map that fixes the infinity, so φ should be a rotation. Then f is conjugate to z^d . So $f^n \sim z^{d^n} \sim g_1^n$ for any $n \in \mathbb{N}^*$. Second, since both f^n and g_1^n have the same degree and the same Julia set which is not a circle, then $f^n = \sigma_n \circ g_1^n$ with σ_n an affine transformation in the symmetry group of the Julia set; see [Be2] for details. So $S_{f^n} = S_{g_1^n}$ for any $n \in \mathbb{N}^*$. And moreover, when Julia set is not a circle, the order of symmetry group of the Julia set is finite; see Lemma 4 in [Be1]. Thus there are only finitely many conjugacy classes which belongs to a strong equivalence class. \square

Notes: the proof of Theorem 1.3 relies on the classification of polynomials with the same Julia set. When do two polynomials have the same Julia set? Historically, commuting polynomials have the same Julia set, as observed by Julia in 1922 [Ju]. Later in 1987, Baker and Eremenko showed when the symmetric group of the Julia set is trivial, polynomials with this Julia set commute; see [BE]. In 1989, Fernández showed there is at most one polynomial with given degree, leading coefficient and Julia set; see [Fe]. Finally in 1992, Beardon showed $\{g | \deg(f) = \deg(g), J(f) = J(g)\} = \{\sigma \circ f | \sigma \in \text{symmetric group of } J(f)\}$; see [Be2].

2.3. Conformal metric of quadratic differential. For meromorphic quadratic differential Q on S , it determines a flat metric $ds^2 = |Q|$, with singularities at zeros and poles of Q .

Trajectories as a foliation. For any meromorphic quadratic differential Q on S , it determines a foliation structure on S with singularities at zeros and poles of Q . A smooth curve on S is a (horizontal) trajectory of Q , if it does not pass through any zero or pole of Q , and for any point p in the curve, the non zero vector dz tangent to this curve satisfy:

$$\arg(Q(p)dz^2) = 0$$

i.e $Q(p)dz^2$ is a positive real number. By a trajectory, we usually mean the trajectory that it is not properly contained in another trajectory, i.e. a maximal trajectory.

Lemma 2.3. *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map with degree $d \geq 2$. Then $S_f dz^2$ is a meromorphic quadratic differential with poles of order two at the critical points of f . For any critical point p of f with order k , near p*

$$S_f(z) = \frac{1 - k^2}{2(z - p)^2} + O\left(\frac{1}{|z - p|}\right)$$

i.e., a neighborhood of p is an infinite cylinder with closed geodesics as trajectories of length $2\pi\sqrt{\frac{k^2-1}{2}}$.

Proof. The only thing we need to show here is that the coefficient of $\frac{1}{(z-p)^2}$ at p is $\frac{1-k^2}{2}$, and for other details of this lemma, please refer to the §6.3 [St]. The coefficient can be verified by a direct computation. \square

2.4. $L_{loc}^{\frac{1}{2}}$ integrability of quadratic differential. For any Riemann surface S , we consider the space $MQ(S)$ of meromorphic quadratic differentials on S . For any $\alpha \in MQ(S)$, we say that it is $L_{loc}^{\frac{1}{2}}$ integrable, if for any point $q \in S$, there is a local coordinate at q , and write α as $h(z)dz^2$ in this coordinate, such that the integration $\int \int \sqrt{|h(z)|} dx dy$ over some neighborhood of p is finite. We say that $\{\alpha_n\} \subset MQ(S)$ $L_{loc}^{\frac{1}{2}}$ -converge to $\alpha \in MQ(S)$, if both α and α_n are $L_{loc}^{\frac{1}{2}}$ integrable, and for any point p , there is some local coordinate at q , and write α and α_n as $h(z)dz^2$ and $h_n(z)dz^2$ in this local coordinate, such that the integral $\int \int \sqrt{|h_n(z) - h(z)|} dx dy$ over some neighborhood of p converges to 0. Actually, the $L_{loc}^{\frac{1}{2}}$ integrable subset of $MQ(S)$ is a vector space.

Any meromorphic quadratic differential $\alpha \in MQ(S)$ with poles of order at most two is $L_{loc}^{\frac{1}{2}}$ integrable.

Lemma 2.4. *For any rational function f with degree $d \geq 2$, $S_{f^n} dz^2$ is $L_{loc}^{\frac{1}{2}}$ integrable.*

Proof. Since for any rational map f , the Schwarzian derivative $S_f dz^2$ has poles of order at most two, then $S_f dz^2$ is $L_{loc}^{\frac{1}{2}}$ integrable, and also $S_{f^n} dz^2$ is $L_{loc}^{\frac{1}{2}}$ integrable.

3. LOCAL CONVERGENCE OF S_{f^n}

In this section, our main goal is to prove Theorem 1.1, the local convergence of the normalized S_{f^n} .

3.1. Bounded Fatou components. In this subsection, we are trying to show that $\lim_{n \rightarrow \infty} \frac{S_{f^n}}{d^{2n}} = 0$ on the bounded Fatou components for any degree $d \geq 2$ polynomial f , which is not conformally conjugate to z^d .

Theorem 3.1. *For any $z \notin \text{Precrit}(f)$ in a bounded Fatou component of a polynomial f with degree $d \geq 2$, which is not conformally conjugate to z^d , then we have:*

$$\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} = 0.$$

Moreover, this is a local uniform convergence.

Proof. First, assume that z is attracted to some fix point z_1 (attracting or parabolic fix point), and z_1 is not a critical point. Then,

$$0 < \lambda = |f'(z_1)| \leq 1$$

For any fixed $0 < \epsilon < 1$, since we have $f^n(z)$ converges to z_1 , there exists $N_o \in \mathbb{N}$ and $M < \infty$, such that for any $n > N_o$, we have:

$$|f'(f^n(z))| \leq 1 + \epsilon \text{ and } |S_f(f^n(z))| < M$$

By (2.2),

$$\begin{aligned} |S_{f^n}(z)| &= \left| \sum_{i=1}^{n-1} S_f \circ f^i(z) \cdot ((f^i)'(z))^2 + S_f(z) \right| \\ &\leq \sum_{i=1}^{n-1} |S_f \circ f^i(z) \cdot ((f^i)'(z))^2| + |S_f(z)| \end{aligned}$$

Since we have

$$|S_f(f^n(z))((f^n)'(z))^2| < M \cdot M_1 \cdot (1 + \epsilon)^{2n}, \text{ for any } n > N_o.$$

where $M_1 = |(f^{N_o})'(z)|$, by the fact that $1 + \epsilon < d$ for $d \geq 2$ and $\epsilon < 1$, it is obvious that $\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} = 0$ is satisfied.

Second assume z is attracted to a critical fix point z_1 . Without loss of generality, we can assume $z_1 = 0$, so $f = az^r + bz^{r+1} + \dots$, with $a \neq 0$ and $2 \leq r \leq d-1$. By Prop. 2.1, we can study the Schwarzian limit at forward iterate of z instead of the Schwarzian limit at z . Then we can assume that z is close to 0. Let's conjugate f to z^r near 0 by a conformal map φ , such that $\varphi(0) = 0$ and $\varphi'(0) \neq 0$,

$$\varphi \circ f \circ \varphi^{-1} = z^r$$

By the cocycle property of Schwarzian and Example 1,

$$\begin{aligned} S_{f^n}(z) &= S_{\varphi^{-1} \circ z^{r^n} \circ \varphi}(z) \\ &= S_{\varphi^{-1}}((\varphi(z))^{r^n}) \cdot ((\varphi^{-1})'(\varphi(z))^{r^n}) \cdot (r^n - 1) \varphi(z)^{r^n-1} \varphi'(z))^2 \\ &\quad + \frac{1 - r^{2n}}{2\varphi(z)^2} \cdot (\varphi'(z))^2 + S_\varphi(z) \end{aligned}$$

Since z is close to 0 and $2 \leq r \leq d-1$, then $\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} = 0$ is obviously true in this case by the above formula.

Third, assume that z is in some Siegel disc \mathbb{D}_o fixed by f . Similar with the previous case, we can move the center of \mathbb{D}_o to 0, and conjugate f by φ on this Siegel disc to a rotation map, i.e.

$$\varphi \circ f = \lambda \cdot \varphi, \text{ with } |\lambda| = 1$$

where we have $\varphi(0) = 0$ and $\varphi'(0) = 1$. And by (2.1),

$$\begin{aligned} S_{f^n}(z) &= S_{\varphi^{-1}(\lambda^n \varphi)}(z) \\ &= S_{\varphi^{-1}(\lambda^n \varphi(z))} \cdot ((\varphi^{-1})'(\lambda^n \varphi(z)))^2 + S_{\varphi}(z) \end{aligned}$$

Since $\lambda^n \varphi(z)$ is in φ 's image of some compact subset of \mathbb{D}_o for any n , then $S_{\varphi^{-1}(\lambda^n \varphi(z))}$ is uniformly bounded. So $\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} = 0$ is obviously satisfied in this case by the above formula.

From above arguments, it is not hard to see the convergence is local uniform. For points in the periodic bounded Fatou components and not in $\text{Precrit}(f)$, we can use similar arguments to show that the result of this theorem is satisfied. And for points attracted to periodic bounded Fatou components, Prop. 2.1 shows that the result is satisfied too. \square

3.2. Basin of infinity. In this subsection, we are going to prove the local convergence of S_{f^n} on the basin of infinity.

Consider the following escape-rate function of a polynomial f with degree $d \geq 2$:

$$G_f(z) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ |x| = \max(\log |x|, 0)$. The escape-rate function $G_f(z)$ is the Green function of the basin of infinity. So it is harmonic on the basin of infinity. Actually, it is a subharmonic function on \mathbb{C} . By taking the partial derivative of $G_f(z)$, we get $g(z) = \frac{\partial G_f(z)}{\partial z}$ is a holomorphic function on the basin of infinity; the zeros of $g(z)$ are exactly the points in $\text{Precrit}(f)$. Moreover, from the definition of $G_f(z)$, it is a limit of harmonic functions converging locally uniformly. The derivatives of this harmonic functions converge. Then we know that the partial derivative commutes with the limit, i.e.

$$\begin{aligned} (3.1) \quad g(z) &= \frac{\partial G_f(z)}{\partial z} = \lim_{n \rightarrow \infty} \frac{\frac{\partial}{\partial z} \log |f^n(z)|}{\partial z} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2d^n} \frac{\partial \log(f^n(z) \bar{f}^n(z))}{\partial z} = \lim_{n \rightarrow \infty} \frac{1}{2} \frac{(f^n(z))'}{d^n f^n(z)} \end{aligned}$$

Theorem 3.2. *For any $z_o \notin \text{Precrit}(f)$ in the basin of infinity of a polynomial f with degree $d \geq 2$, we have:*

$$\lim_{n \rightarrow \infty} \frac{S_{f^n}(z_o)}{d^{2n}} = -2(g(z_o))^2 = -2 \left(\frac{\partial G_f(z_o)}{\partial z} \right)^2$$

Moreover, this is a local uniform convergence.

Proof: Since $g(z) = 0$ if and only if $z \in \text{Precrit}(f)$, then we have $g(z_o) \neq 0$, because $z_o \notin \text{Precrit}(f)$.

Let $r_n = \frac{(f^n(z_o))'}{2g(z_o)d^n f^n(z_o)}$, by (3.1), we get

$$\lim_{n \rightarrow \infty} r_n = 1$$

Moreover, let $f(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0$ with $a_d \neq 0$, and an easy calculation shows that

$$S_f(z) = \frac{1-d^2}{2z^2} h(z), \text{ and } \lim_{z \rightarrow \infty} h(z) = 1,$$

where $h(z)$ is a rational function. Let $s_n = h(f^n(z_o))$, since $\lim_{n \rightarrow \infty} f^n(z_o) = \infty$, so we get $\lim_{n \rightarrow \infty} s_n = 1$. Then

$$\begin{aligned} S_f(f^n(z_o))((f^n)'(z_o))^2 &= \frac{(1-d^2)s_n((f^n)'(z_o))^2}{2(f^n(z_o))^2} \\ &= 2(1-d^2)s_n r_n^2 (g(z_o))^2 d^{2n}, \end{aligned}$$

Substituting the above formula into (2.2), we get

$$\begin{aligned} \frac{S_{f^n}(z_o)}{d^{2n}} &= \frac{\sum_{i=1}^{n-1} S_f(f^i(z_o))((f^i)'(z_o))^2 + S_f(z_o)}{d^{2n}} \\ &= 2(g(z_o))^2 (1-d^2) \frac{\sum_{i=0}^{n-1} s_i r_i^2 d^{2i}}{d^{2n}} \end{aligned}$$

Because both r_n and s_n converge to 1 as n tends to ∞ , it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_{f^n}(z_o)}{d^{2n}} &= 2(g(z_o))^2 (1-d^2) \lim_{n \rightarrow \infty} \frac{1-d^{2n}}{(1-d^2)d^{2n}} \\ &= -2(g(z_o))^2 = -2 \left(\frac{\partial G_f(z_o)}{\partial z} \right)^2 \end{aligned}$$

The fact that this convergence is local uniform can be deduced from the fact that both $r_n(z)$ and $s_n(z)$ converge locally uniformly. \square

Remark. Alternately, the result of Theorem 3.2 can be seen in the language of ‘‘Schwarzian between conformal metrics’’. On the complex plane \mathbb{C} , we define:

$$\widehat{S}(e^{\sigma_1}|dz|^2, e^{\sigma_2}|dz|^2) = \left(\sigma_{1zz} - \frac{1}{2}\sigma_{1z}^2 - (\sigma_{2zz} - \frac{1}{2}\sigma_{2z}^2) \right) dz^2,$$

where $\sigma_z = \frac{\partial \sigma}{\partial z}$. Easy to know, we have

- $S_f dz^2 = \widehat{S}(f^*|dz|^2, |dz|^2)$, for any non constant holomorphic map f .
- $\widehat{S}(c_1 \rho_1 |dz|^2, c_2 \rho_2 |dz|^2) = \widehat{S}(\rho_1 |dz|^2, \rho_2 |dz|^2)$, for any positive constant numbers c_1 and c_2 .
- $\widehat{S}(\rho_1 |dz|^2, \rho_3 |dz|^2) = \widehat{S}(\rho_1 |dz|^2, \rho_2 |dz|^2) + \widehat{S}(\rho_2 |dz|^2, \rho_3 |dz|^2)$.

Then we have

$$\begin{array}{ccc}
\frac{1}{d^{2n}} \left(\widehat{S}(f^{n*}|dz|^2, |dz|^2) + \widehat{S}(|dz|^2, f^{n*}|\frac{dz}{z}|^2) \right) & = & \frac{1}{d^{2n}} S_{f^n} dz^2 + \frac{1}{d^{2n}} \widehat{S}(|dz|^2, \frac{1}{d^{2n}} f^{n*} |\frac{dz}{z}|^2) \\
\parallel & \searrow n \rightarrow \infty & \downarrow 0 \quad n \rightarrow \infty \\
\frac{1}{d^{2n}} \widehat{S}(f^{n*}|dz|^2, f^{n*}|\frac{dz}{z}|^2) & = & \frac{1}{d^{2n}} f^{n*} \widehat{S}(|dz|^2, |\frac{dz}{z}|^2) = \frac{1}{d^{2n}} f^{n*} \left(\frac{-dz^2}{2z^2} \right) \rightarrow -2 \left(\frac{\partial G_f(z)}{\partial z} \right)^2 dz^2
\end{array}$$

For more about the Schwarzian of conformally equivalent Riemannian metrics, please refer to [OS].

Nonlinearity. Similar with the Scharzian derivative, we can define nonlinearity N_f of a nonconstant holomorphic function f on the complex plane.

$$N_f = \frac{f''}{f'}$$

Nonlinearity $N_f \equiv 0$ if and only if f is an affine transformation. Sometimes we can view N_f as a one form $N_f dz$. We have the following cocycle property:

$$N_{f \circ g} dz = g^*(N_f dz) + N_g dz$$

Moreover, $N_f dz$ has a pole of order one at critical point of f . Using the same argument in Theorem 1.1, we have

Theorem 3.3. *Let f be an polynomial with degree $d \geq 2$ and not conformally conjugate to z^d , and X_o be its basin of infinity. Then we have:*

- $\lim_{n \rightarrow \infty} \frac{(f^n)' dz}{d^n f^n} = \partial G_f$, on X_o .
- $\lim_{n \rightarrow \infty} \frac{N_{f^n} dz}{d^n} = \partial G_f$, on $\mathbb{C} \setminus \overline{\text{Precrit}(f)}$.
- $\lim_{n \rightarrow \infty} \frac{(f^n)''' dz^2}{d^{2n} (f^n)'} = -\frac{(\partial G_f)^2}{2}$, on $\mathbb{C} \setminus \overline{\text{Precrit}(f)}$.

In each case, the convergence is locally uniform.

Proof of Theorem 1.1. Since we have $G_f \equiv 0$ on the bounded Fatou components of f , then this theorem is an easy consequence of Theorem 3.1 and Theorem 3.2. \square

3.3. Global convergence on the Fatou set. Recall the definition of $L_{loc}^{\frac{1}{2}}$ integrability and $L_{loc}^{\frac{1}{2}}$ convergence of meromorphic quadratic differentials on the Riemann surface in §2.2. For any rational function f with degree $d \geq 2$, by Lemma 2.4, $S_{f^n} dz^2$ is $L_{loc}^{\frac{1}{2}}$ integrable.

Corollary 3.4. *Let $S_{f^n} dz^2$ be the meromorphic quadratic differential on \mathbb{P}^1 determined by the Schwarzian derivative of f^n , where f is a polynomial with degree $d \geq 2$ and not conformally conjugate to z^d . Then*

$$\lim_{n \rightarrow \infty} \frac{S_{f^n}(z)}{d^{2n}} dz^2 = -2 \left(\frac{\partial G_f(z)}{\partial z} \right)^2 dz^2,$$

on the Fatou set (including ∞) of f , converging in the sense of $L_{loc}^{\frac{1}{2}}$.

Proof. The proof of this corollary follows easily from the arguments in the proof of Theorem 1.1, together with the triangle inequality. So we omit the details here. \square

4. GEOMETRIC LIMIT OF METRIC SPACES

In this section, we discuss the possible limit of metric spaces coming from Schwarzian derivatives.

4.1. Ultrafilter and ultralimit. A non-principal ultrafilter ω is a set consisting of a collection of subsets of \mathbb{N} , satisfying

- If $A \subset B \subset \mathbb{N}$ and $A \in \omega$, then $B \in \omega$.
- For any disjoint union $\mathbb{N} = A_1 \cup A_2 \cdots \cup A_n$, there exists one and only one $A_i \in \omega$.
- For any finite set $A \subset \mathbb{N}$, $A \notin \omega$,

We can view a non-principal ultrafilter ω as a finitely additive measure on \mathbb{N} , only taking values in $\{0, 1\}$, where any finite subset of \mathbb{N} has measure zero and \mathbb{N} has measure 1. By Zorn's lemma there exists some non-principal ultrafilter, and non-principal ultrafilter on \mathbb{N} is not unique. Hereafter, we fix a non-principal ultrafilter ω . For more details about the ultrafilter, please refer to Chapter I §5 [BH].

Let Y be a compact Hausdorff space. For any sequence of points $\{y_i\}_{i=1}^\infty \subset Y$, there is an unique point $y_o \in Y$ such that $\{i | y_i \in U\} \in \omega$ for any open set U containing y_o . This $y_o = \lim_\omega y_i$ is denoted as the ultralimit of $\{y_i\}_{i=1}^\infty$.

Let $\{(Y_n, d_n, b_n)\}_{n=1}^\infty$ be a sequence of pointed metric spaces. Let \tilde{Y}_ω be the set of all the sequences (y_n) with $y_n \in Y_n$ satisfying:

$$\lim_\omega d_n(y_n, b_n) < \infty;$$

here, the ultralimit is taken in the space of $[0, +\infty]$. Set

$$\tilde{d}_\omega((x_n), (y_n)) := \lim_\omega d_n(x_n, y_n) < \infty,$$

with (x_n) and $(y_n) \in \tilde{Y}_\omega$. This is a pseudo-distance on \tilde{Y}_ω . Let $(Y_\omega, d_\omega, b_\omega) := (\tilde{Y}_\omega, \tilde{d}_\omega, (b_n))/\sim$, where we identify points with zero \tilde{d}_ω -distance. The point metric space $(Y_\omega, d_\omega, b_\omega)$ is called the ultralimit of $\{(Y_n, d_n, b_n)\}_{n=1}^\infty$; see Chapter I §5 [BH].

4.2. The ultralimit of the Schwarzian metrics. Let f be a polynomial with degree $d \geq 2$. The Schwarzian derivative S_{f^n} determines a metric space (X_n, d_n) , where $X_n = \mathbb{C} \setminus \text{Crit}(f^n)$ and d_n is the arc length metric $ds = \sqrt{\left| \frac{S_{f^n}}{d^{2n}} \right|} |dz|$. This is a complete geodesic space with non positive curvature. Fix a point $a \notin \text{Precrit}(f)$ on the basin of infinity. In this section we are considering the ultralimit $(X_\omega, d_\omega, a_\omega)$ of the pointed metric spaces (X_n, d_n, a) .

Proposition 4.1. *$(X_\omega, d_\omega, a_\omega)$ is a complete geodesic space.*

Proof: The metric space $(X_\omega, d_\omega, a_\omega)$ is complete, since the ultralimit of any sequence of pointed metric spaces is complete; see §1 Lemma 5.53 [BH]. Moreover, the geodesic property is also preserved by passing to the ultralimit. The pointed metric space $(X_\omega, d_\omega, a_\omega)$ is a geodesic space. Indeed, $\{(X_n, d_n, a)\}$ are pointed geodesic spaces. For any two points $x_\omega = (x_i)$ and $y_\omega = (y_i)$ in X_ω , there is (z_i) such that $d_i(x_i, z_i) = d_i(z_i, y_i) = \frac{1}{2}d_i(x_i, y_i)$. From this fact, we know that $z_\omega = (z_i) \in X_\omega$ and also

$$\lim_\omega \frac{1}{2}d_i(x_i, y_i) = \lim_\omega d_i(x_i, z_i) = \lim_\omega d_i(z_i, y_i),$$

i.e., $d_\omega(x_\omega, z_\omega) = d_\omega(z_\omega, y_\omega) = \frac{1}{2}d_\omega(x_\omega, y_\omega)$. So x_ω and y_ω have a midpoint in X_ω . Which means X_ω is a geodesic space, since it is complete. \square

4.3. The flat metric on the basin of infinity. Let X_o be the basin of infinity of f and d_o be the arc length metric from $ds = \sqrt{2} \left| \frac{\partial G_f}{\partial z} \right| |dz|$; see Figure 1.1. The metric space (X_o, d_o) is not complete. If the filled Julia set $K(f)$ of f is disconnected, then we can complete X_o as follows

Lemma 4.2. *For polynomial f with degree $d \geq 2$ and $K(f)$ disconnected, the metric completion (\overline{X}_o, d_o) of (X_o, d_o) is a quotient of \mathbb{C} , obtained by collapsing each connected component of $K(f)$ to a point. The completion \overline{X}_o is homeomorphic to \mathbb{R}^2 . Moreover, (\overline{X}_o, d_o) is a geodesic space.*

Proof. Since $K(f)$ is disconnected, the first two conclusions of this lemma follow immediately from the flat structure of the metric d_o ; compare [DM]. A complete metric space is geodesic if there exists a midpoint for any two points on this metric space. It is obvious that (\overline{X}_o, d_o) has this property. So it is a geodesic space. \square

Let E be the set of $\overline{X}_o \setminus X_o$. The set E is totally disconnected. So we call E the *ends* of X_o . Each point $e \in E$ corresponds to a connected component of $K(f)$. Let $C = X_o \cap \text{Precrit}(f)$. For any *end* in E , there is a sequence of annuli on $X_o \setminus C$ such that they nest down to this *end* with d_o -diameters tending to zero; see the flat metric structure of X_o in [DM]. The following lemma follows easily from Theorem 1.1.

Lemma 4.3. *For any piece wise smooth and compact curve $\gamma \subset X_o \setminus C$, the d_n -length $d_n(\gamma)$ converge to the d_o -length $d_o(\gamma)$. Moreover, for any point $p \in X_o \setminus C$, there is a small neighborhood $U \subset X_o \setminus C$ of p , such that $\lim_{n \rightarrow \infty} d_n(x, y) = d_o(x, y)$, uniformly for $x, y \in U$.*

The following proposition is an essential ingredient in the proof of Theorem 1.1.

Proposition 4.4. *Let f be a polynomial with degree $d \geq 2$ and $J(f)$ disconnected. For any geodesic metric \tilde{d} on $\overline{X_o}$ with (X_o, \tilde{d}) locally isometric to (X_o, d_o) under the identity map, we have $\tilde{d} = d_o$ on $\overline{X_o}$.*

To prove this lemma, we need the tree structure of (X_o, d_o) ; see [DM]. Specifically, this is a quotient $\pi : X_o \rightarrow T(f)$, defined by collapsing each connected component of the level set of $G_f(z)$ to a point. There is a canonical map F on $T(f)$ induced from f .

$$\begin{array}{ccc} X_o & \xrightarrow{f} & X_o \\ \pi \downarrow & & \downarrow \pi \\ T(f) & \xrightarrow{F} & T(f) \end{array}$$

The space $T(f)$ has a simplicial structure, and is equipped with a metric from G_f . The set of vertices is $V = \cup_{n, m \in \mathbb{Z}} F^m(F^n(\text{branch points}))$. The distance between two points in the same edge is given by the difference of their G_f values. Let $S = \pi^{-1}(V)$. Then $X_o \setminus S$ consists of countably many connected components. Each of them is an annulus. So we view $X_o \setminus S$ as a set of annuli. The map π is a one to one map from the annuli to the edges of $T(f)$. For any annulus $A \in X_o \setminus S$, it is a cylinder with finite height in the d_o -metric.

The height of A , denoted as $H(A)$, is equal to the length of the edge $\pi(A)$. Also, we define the level of A as $L(A)$ to be the G_f value of the middle point of $\pi(A)$. The closed geodesics of A are the level sets of G_f . They have the same arc length in the d_o -metric, denoted as $C(A)$.

The map f sends annulus in $X_o \setminus S$ to annulus. For each annulus $A \in X_o \setminus S$, there is a well defined local degree d_A , defined as the topological degree of $f|_A$. Moreover, let $N(f) = \max\{G_f(c) \mid c \text{ is a critical point of } f\}$. We have the following properties:

•

$$(4.1) \quad \sum_{B \in X_o \setminus S, L(B)=L(A)} C(B) = \sqrt{2}\pi$$

•

$$(4.2) \quad L(f(A)) = d \cdot L(A), \quad H(f(A)) = d \cdot H(A), \quad C(f(A)) = \frac{d_A}{d} C(A)$$

- d_A is equal to one plus the number of critical points (counted with multiplicity) enclosed in A .
- The points enclosed in A should have G_f value less than $L(A)$. If $L(A) < N(f)$, then the annulus A can not enclose the critical point(s) with G_f value equals $N(f)$. And because f has $d-1$ critical points counted with multiplicity, then $d_A \leq d-1$ when $L(A) < N(f)$. From (4.1) and (4.2), for any annulus A satisfying $d^n L(A) < N(f)$, we have $L(f^i(A)) = d^i L(A) < N(f)$, i.e. $d_{f^i(A)} \leq d-1$ for any $1 \leq i \leq n$. Consequently,

$$(4.3) \quad C(A) \leq \left(\frac{d-1}{d}\right)^n \cdot C(f^n(A)) \leq \left(\frac{d-1}{d}\right)^n \cdot \sqrt{2}\pi$$

- Let $|f^{-1}(A)|$ be the number of annuli in the set $f^{-1}(A)$. We have $|f^{-1}(A)|$ equals to d minus the total number of critical points (counted with multiplicity) enclosed in the annuli in $f^{-1}(A)$.
- Let $A \in X_o \setminus S$. Any point $x \in \overline{X_o}$ with $G_f(x) \leq L(A)/d$ is enclosed in one of the annulus $B \in X_o \setminus S$ with $L(B) = L(A)$.
- Any two annuli $A, B \in X_o \setminus S$ with $L(A) = L(B)$ have the same height.

For more details about these annuli and the tree structure of the basin of infinity, please see [DM].

Proof of Proposition 4.4. Since X_o is dense in $\overline{X_o}$ for both the metrics \tilde{d} and d_o , then it suffices to prove that $d_o(x, y) = \tilde{d}(x, y)$ for any two points $x, y \in X_o$.

One direction: $d_o(x, y) \geq \tilde{d}(x, y)$. From the definition of d_o , for any $\epsilon > 0$, there is an arc $\gamma \subset X_o$ connecting x and y with d_o length $d_o(\gamma) < d_o(x, y) + \epsilon$. Since these two metrics are locally isometric on X_o , for the length $\tilde{d}(\gamma)$ of γ in the \tilde{d} -metric, we have

$$\tilde{d}(x, y) \leq \tilde{d}(\gamma) = d_o(\gamma) < d_o(x, y) + \epsilon$$

Let $\epsilon \rightarrow 0$, then we get $\tilde{d}(x, y) \leq d_o(x, y)$.

The other direction: $d_o(x, y) \leq \tilde{d}(x, y)$. Choose a geodesic $\tilde{g} : [0, r] \mapsto \overline{X_o}$ in the \tilde{d} -metric, with $\tilde{g}(0) = x$ and $\tilde{g}(r) = y$. Our goal is to construct an arc $\gamma \subset X_o$ from \tilde{g} , connecting x and y with d_o length $d_o(\gamma) < r + \epsilon$ for any fixed $\epsilon > 0$.

From (4.3), there is some small level $l > 0$ such that the set of annuli $T = \{A \in X_o \setminus S \mid L(A) = l\}$ is not empty and $C(A) < \frac{\epsilon}{3}$ for any $A \in T$. Since all the annuli with the same level have the same height, so $h = H(A)$ for $A \in T$ is well defined. We order the elements of T as A_1, A_2, \dots, A_k , with $C(A_i) \leq C(A_j)$ for any $1 \leq i < j \leq k$. Let $T_s = f^{-s}(T)$ and decompose T_s into T_s^- and T_s^+ , with T_s^+ the set of annuli crossed by \tilde{g} . So the length of \tilde{g} in each annulus belonging to T_s^+ is at least the height of this annulus. Let $|\cdot|$ be the number of elements of a set. Then we have:

$$\sum_{s \in \mathbb{N}} |T_s^+| \cdot \frac{h}{d^s} \leq \tilde{d}\text{-length of } \tilde{g} = r < \infty$$

Form the above formula, there is some big n , such that $|T_n^+| \cdot \frac{h}{d^n} < h$, i.e., $|T_n^+| < d^n$. As $|f^{-1}(A)|$ is equal to d minus the total number of critical points (counted with multiplicity) enclosed in the annuli belonging to $f^{-1}(A)$, and f has $d - 1$ critical points, then

$$|T_{s+1}| = |f^{-1}(T_s)| > d \cdot |T_s| - d, \quad |f^{-n}(A)| \leq d^n$$

since any two annuli with the same level won't enclose each other, i.e., they won't share a common critical point inside. Consequently,

$$|T_n| > d^n \cdot |T| - d^n - d^{n-1} - \dots - d > d^n \cdot k - 2d^n$$

Combining the above formula with $|T_n^+| < d^n$, we have $|T_n^-| = |T_n| - |T_n^+| > d^n \cdot (k-3)$. From (4.2), for any $1 \leq i \leq k$,

$$|f^{-n}(A_i)| \leq d^n, \quad \frac{C(A_i)}{d^n} \leq C(B) \text{ any } B \in f^{-n}(A_i)$$

We order T_n as $B_1, B_2, \dots, B_{|T_n|}$, such that $\beta_j = C(B_j) \leq \beta_{j+1} = C(B_{j+1})$ for any $1 \leq j \leq |T_n| - 1$. From the above formula, we have

$$\beta_{d^n \cdot (i-1) + j} \geq \frac{C(A_i)}{d^n}, \quad 1 \leq i \leq k-3 \text{ and } 1 \leq j \leq d^n,$$

Then we get,

$$\begin{aligned} \sum_{B \in T_n^+} C(B) &= \sum_{B \in T_n} C(B) - \sum_{B \in T_n^-} C(B) \\ &\leq \sqrt{2}\pi - \sum_{i=1}^{k-3} \sum_{j=1}^{d^n} \beta_{d^n \cdot (i-1) + j} \\ &\leq \sum_{i=1}^k C(A_i) - \sum_{i=1}^{k-3} d^n \cdot \frac{C(A_i)}{d^n} \\ &= C(A_{k-2}) + C(A_{k-1}) + C(A_k) \\ &< \epsilon. \end{aligned}$$

Now, we can construct an arc $\gamma \subset X_o$ from \tilde{g} as follows (without loss of generality, we can always assume that both $G_f(x)$ and $G_f(y)$ are greater than l , and $n > 1$):

- Choose a minimal $r_1 \in [0, r]$ such that $\tilde{g}(r_1)$ lying on the outer boundary of some annulus $B^1 \in T_n^+$.
- Choose a maximal $r'_1 \in [0, r]$ such that $\tilde{g}(r'_1)$ lying on the outer boundary of the annulus B^1 .
- Replace the arc $\tilde{g}([r_1, r'_1]) \subset \tilde{g}$ with a shortest curve γ_1 on the outer boundary of B^1 connecting $\tilde{g}(r_1)$ and $\tilde{g}(r'_1)$.
- Do the same thing as the previous three steps, we can find a minimal r_2 and maximal $r'_2 \in [r'_1, r]$, such that $\tilde{g}(r_2)$ and $\tilde{g}(r'_2)$ lying on the outer boundary of some $B^2 \in T_n^+$ and $r'_2 - r_2 > 0$. Replace $\tilde{g}([r_2, r'_2]) \subset \tilde{g}$ with a shortest curve γ_2 on the outer boundary of B^2 connecting $\tilde{g}(r_2)$ and $\tilde{g}(r'_2)$.
- Keep doing the same thing as previous step by step, we can replace sub-arcs of \tilde{g} by arcs γ_i on the boundary of $B^i \in T_n^+$. This process will stop under finite steps, since $B^i \neq B^j$ for any $i < j$ and $|T_n^+| < \infty$.

From the above construction, we get a new arc γ from \tilde{g} . We have $\gamma \subset X_o$. In fact, for any point $p \in \gamma$, p is not enclosed by any annulus $B \in T_n$. So we have $G_f(p) > \frac{h}{d^{n+1}}$. And

$$\begin{aligned} d_o(x, y) &\leq d_o(\gamma) \leq r + d_o(\gamma_1) + d_o(\gamma_2) + \cdots \\ &\leq r + C(B^1) + C(B^2) + \cdots \\ &\leq r + \sum_{B \in T_n^+} C(B) < r + \epsilon. \end{aligned}$$

Let $\epsilon \rightarrow 0$, we get $d_o(x, y) \leq r = \tilde{d}(x, y)$. \square

Proof of Theorem 1.2. First, we construct a natural map ρ

$$\rho : X_o \setminus C \mapsto X_\omega,$$

as: for any $p \in X_o \setminus C$, $\rho(p) = p_\omega = (p_n = p) \in X_\omega$. This map is well defined, since $\{d_n(p, a)\}$ is uniformly bounded by Lemma 4.3. Let p, q be two points in $X_o \setminus C$, and p_ω, q_ω the ρ image of p, q in X_ω . We want to show that $d_\omega(p_\omega, q_\omega) \leq d_o(p, q)$. For any $\epsilon > 0$, we can choose a smooth curve on the basin of infinity with d_o -length less than $d_o(p, q) + \epsilon$. By a small perturbation, we can assume this curve does not pass any point in C , and this curve is also on the basin of infinity. Since this curve is compact, by Lemma 4.3, the d_n -length of this curve converges to the d_o -length of this curve. So for n big enough, the d_n -length of this curve is less than $d_o(p, q) + 2\epsilon$. Then $d_\omega(p_\omega, q_\omega) \leq d_o(p, q) + 2\epsilon$. Let $\epsilon \rightarrow 0$, we get $d_\omega(p_\omega, q_\omega) \leq d_o(p, q)$. Also, from Lemma 4.3, this is a locally isometric and distance non-increasing embedding. Then we can extend the map ρ from $X_o \setminus C$ to $\overline{X_o}$:

$$\rho : \overline{X_o} \longrightarrow X_\omega$$

For any end $e \in E$, let K_e be the corresponding connected component of $K(f)$ and $X_\omega^e = \{(x_i) \in X_\omega \mid \lim_\omega x_i \in K_e\}$. And for any $c \in C$, let $X_\omega^c = \{(x_i) \in X_\omega \mid \lim_\omega x_i = c\}$. Obviously, the set

$$X_\omega = (\cup_{\alpha \in C \cup E} X_\omega^\alpha) \cup \rho(X_o \setminus C),$$

is a disjoint union, i.e.

$$X_\omega^{\alpha_1} \cap X_\omega^{\alpha_2} = \emptyset \text{ and } X_\omega^{\alpha_1} \cap \rho(X_o \setminus C) = \emptyset, \text{ for any } \alpha_1 \neq \alpha_2 \in C \cup E$$

Indeed, there is a closed annulus A in $X_o \setminus C$ with the corresponding parts of α_1 and α_2 in the two different components of $\mathbb{C} \setminus A$. Let h be the distance between the two boundaries of A in the d_o -metric. we have $h > 0$. Since any arc connecting two points in distinct components of $\mathbb{C} \setminus A$ should across A . By Lemma 4.3, the d_n -distance of the two boundaries of A converge to h . So we have $d_\omega(X_\omega^{\alpha_1}, X_\omega^{\alpha_2}) \geq h > 0$. Consequently, $X_\omega^{\alpha_1} \cap X_\omega^{\alpha_2} = \emptyset$. Similarly, any point $x_\omega \in \rho(X_o \setminus C)$, we have $d_\omega(X_\omega^{\alpha_1}, x_\omega) > 0$, then $X_\omega^{\alpha_1} \cap \rho(X_o \setminus C) = \emptyset$.

From above, for any two distinct points α, β in $C \cup E$ and $p_\omega \in \rho(X_o \setminus C)$, we have $d_\omega(\rho(\alpha), X_\omega^\beta) > 0$ and $d_\omega(\rho(\alpha), p_\omega) > 0$. Moreover, since $\overline{X_o}$ is homeomorphic to \mathbb{R}^2

by Lemma 4.2, so $\rho : \overline{X}_o \mapsto \rho(\overline{X}_o)$ is a distance non-increasing homeomorphism. We want to show $(\rho(\overline{X}_o), d_\omega)$ is a geodesic space and ρ is locally isometric at the points in C . Then by Proposition 4.4, we may conclude that ρ is an isometric embedding.

For any $x_\omega \in X_\omega^\alpha$ and $y_\omega \notin X_\omega^\alpha$, with $\alpha \in C \cup E$, we want to show any geodesic g_ω connecting x_ω and y_ω should pass through $\rho(\alpha)$. For any closed annulus $A \subset X_o \setminus C$ with the points $\lim_\omega(x_i)$ and $\lim_\omega(y_i)$ lying in the different components of $\mathbb{C} \setminus A$, as previous, $X_\omega \setminus \rho(A)$ consists two connected components, with distance at least the distance of the two boundaries of A in the d_o -metric. And since x_ω and y_ω are in different components of $X_\omega \setminus \rho(A)$, the geodesic connecting them should intersect $\rho(A)$. Let $\{A_i\}_{i=1}^\infty$ be a sequence of such annuli, and they nest down to α in the sense that $\lim_{i \rightarrow \infty} \text{diameter}(\alpha \cup A_i) = 0$ in the d_o -metric. Choose some x_ω^i in $\rho(A_i) \cap g_\omega$. Since $\{A_i\}_{i=1}^\infty$ nest down to α and the map ρ does not increase the distance, then we have that $\{x_\omega^i\}$ converges to $\rho(\alpha)$. As the geodesic is compact, we know that $\rho(\alpha)$ should in the geodesic g_ω .

If there is a geodesic $g_\omega \subset X_\omega$ with two end points in $\rho(\overline{X}_o)$ such that it has some point $p_\omega \in g_\omega \cap (X_\omega \setminus \rho(\overline{X}_o))$. Assume p_ω belongs to X_ω^α with $\alpha \in C \cup E$. Then p_ω divides g_ω in to two parts. Each of these two parts should pass through $\rho(\alpha)$. Which means g_ω can not be the shortest curve connecting the two end points. In all, any geodesic with two ends in $\rho(\overline{X}_o)$ should be contained in $\rho(\overline{X}_o)$. So $(\rho(\overline{X}_o), d_\omega)$ is a geodesic space, since X_ω is a geodesic space. Plus $\rho|_{X_o \setminus C}$ is locally isometric and distance non-increasing embedding, we get $\rho|_{X_o}$ is locally isometric.

For any $\alpha \in E \cup C$, since there always exists a sequence of $\{c_i\}_{i=1}^\infty \subset C$ converging to α , then for any $l > 0$ big enough, we can always choose a sequence of points $\{x_i\}_{i=1}^\infty \subset X_o \setminus C$, such that $d_i(x_i, a_i) = l$ and $\lim_\omega x_i$ in α 's corresponding subset of \mathbb{C} . Then we have $x_\omega = (x_i) \in X_\omega^\alpha$ and $d_\omega(x_\omega, a_\omega) = l$.

For any $c \in C$, to prove that X_ω^c is a real ray, it suffices to prove that for any two sequence $\{x_n\}$ and $\{y_n\}$ converging to c , with

$$\lim_{n \rightarrow \infty} [d_n(x_n, a) = d_n(y_n, a)] = l > l_o = d_o(a, c)$$

then, we have $\lim_{n \rightarrow \infty} d_n(x_n, y_n) = 0$. This follows easily from Lemma 4.7 proved below. \square

4.4. Real rays attached to C . In this subsection, we are going to complete the proof of Theorem 1.2 by showing Lemma 4.7. What remains is to show that the extra pieces of X_ω are real rays attached to X_o . The basic idea is to show that X_n has no “bulb” near the critical points of f^n . For doing this, we need to use the fact that X_n is a metric space with non-positive curvature.

Let S be a closed Riemann surface with genus $g \geq 2$ and Q be a holomorphic quadratic differential on S . Then Q determines a flat metric $ds^2 = |Q|$ with finite singularities on S at zeros of Q . At non-singular points, it is flat, so it has curvature

0; at singular points, it's a cone with angle $k\pi$ for $3 \leq k \in \mathbb{N}$. So at the singular points, it has negative curvature. Then in this metric, S is a complete arc length metric space with non-positive curvature. For the definition and properties of the curvature, please refer to p. 159 [BH]. Lift this metric to the universal cover \tilde{S} of S , we also get a metric on \tilde{S} with non-positive curvature.

Lemma 4.5. *\tilde{S} is a complete CAT(0) unique geodesic space, any geodesic locally is a straight line at non singular point.*

Proof. By [Ah], \tilde{S} is an unique geodesic space (any two points are connected by an unique geodesic) with geodesic locally straight line, and \tilde{S} is also complete. Moreover, since it is a complete and simply connected metric space with non positive curvature, by Cartan-Hadamard Theorem, such space is a CAT(0) space; see p. 193 [BH]. \square

Fix a point $c \in C$ and some very small $\epsilon > 0$. Let $c_\epsilon \subset X_o \setminus C$ be the closed curve at ϵ -distance from c in the d_o -metric. For each n , choose some closed geodesic (in the d_n -metric) $c_n \subset X_n$ such that c_n is sufficiently close to c ; see Lemma 2.3. By Lemma 4.3 and Lemma 2.3, there is some $M < \infty$, such that the d_n -length of c_ϵ $d_n(c_\epsilon) < M \cdot \epsilon$ and $d_n(c_n) < M/d^n$. Choose $x_n \in c_\epsilon$ and $y_n \in c_n$, such that $r_n = d_n(x_n, y_n) = d_n(c_\epsilon, c_n)$. We can do this is because both c_ϵ and c_n are compact. Choose a geodesic $g_n : [0, r_n] \mapsto X_n$ with $g_n(0) = x_n$ and $g_n(r_n) = y_n$. Let A_n be the open annulus bounded by c_ϵ and c_n . We have that $g_n((0, r_n)) \subset A_n$. Otherwise it won't be the shortest curve connecting the two boundaries of A_n .

We can construct closed Riemann surface S_n with genus $g \geq 2$ from X_n . The metric space (X_n, d_n) has finitely many infinite cylinders (cylinder with infinite height). Each such infinite cylinder lying in some neighborhood of a critical point (including $\{\infty\}$) of f^n . So cut the infinite cylinders off X_n along some of the closed geodesics inside the cylinders, such that the closed geodesic is much closer than c_n to the critical point. Then we get X'_n . Double X'_n , and glue them together along the corresponding boundaries to get a closed surface S_n . The metric on S_n is the obvious metric induced from X'_n . Consider the universal cover \tilde{S}_n of S_n with the induced metric \tilde{d}_n from S_n . Topologically, \tilde{S}_n is a unit disk. Let $\tilde{A}_n \subset \tilde{S}_n$ be one of the connected components of the preimage of $A_n \subset X'_n$. Then \tilde{A}_n is a strip on \tilde{S}_n separating \tilde{S}_n into two connected components. Since the projection of any curve connecting this two components should be some curve across $A_n \subset X'_n$, the distance of these two components is $d_n(x_n, y_n)$ obtained by some lift \tilde{g}_n of g_n connecting these two components.

The boundaries $\partial\tilde{A}_n$ of \tilde{A}_n are two curves in the preimages of $c_\epsilon, c_n \subset X'_n$. The preimage of g_n on \tilde{A}_n cuts \tilde{A}_n into quadrilaterals. All of them can be mapped into each other by some isometry of \tilde{S}_n . Pick one of these quadrilateral \tilde{B}_n with $\tilde{g}_n \subset \partial\tilde{B}_n$. Then \tilde{B}_n is a copy of the lift of $A_n \setminus g_n$. Denote \tilde{c}_ϵ and \tilde{c}_n as the lift of c_ϵ and c_n on $\partial\tilde{B}_n$, and \tilde{g}_n^o the other lift of g_n on $\partial\tilde{B}_n$.

Lemma 4.6. *For any two points $\tilde{z}_1 \in \tilde{c}_\epsilon$ and $\tilde{z}_2 \in \tilde{c}_n$, there is an unique geodesic \tilde{g} connecting these two points, and if we varies \tilde{z}_1 and \tilde{z}_2 continuously, then \tilde{g} varies continuously. In particular, any point $\tilde{q} \in \tilde{B}_n$, there is some geodesic \tilde{g}_q passing though \tilde{q} with two ends in \tilde{c}_ϵ and \tilde{c}_n .*

Proof. By Lemma 4.5, there is an unique geodesic \tilde{g} connecting \tilde{z}_1 and \tilde{z}_2 . Because \tilde{S}_n is a complete and simply connected CAT(0) metric space, by Cartan-Hadamard theorem in p. 193 [BH], geodesic varies continually with respect to the two end points.

Assume there is some point $\tilde{q} \in \tilde{B}_n$ such that any geodesic with two ends in \tilde{c}_ϵ and \tilde{c}_n won't pass though it. Choose $\tilde{z}_1(t) \in \tilde{c}_\epsilon$ and $\tilde{z}_2(t) \in \tilde{c}_n$ varies from the ends of \tilde{g}_n to the ends of \tilde{g}_n^o . Then the corresponding geodesics varies from \tilde{g}_n to \tilde{g}_n^o without touching \tilde{q} . From this we get that, in $\tilde{S}_n \setminus \tilde{q}$, $\partial \tilde{B}_n$ is homotopic to a point. This is impossible since \tilde{S}_n is topologically a disc. \square

Lemma 4.7. *Let \tilde{q}_1 and \tilde{q}_2 be two points in $\tilde{B}_n \setminus (\tilde{c}_n \cup \tilde{c}_\epsilon)$, with $l_1 = \tilde{d}_n(\tilde{q}_1, \tilde{c}_\epsilon)$ and $l_2 = \tilde{d}_n(\tilde{q}_2, \tilde{c}_\epsilon)$. Then $\tilde{d}_n(\tilde{q}_1, \tilde{q}_2) \leq |l_1 - l_2| + 9r_3 + 7r_4$, where r_3 and r_4 are \tilde{d}_n -lengths of \tilde{c}_ϵ and \tilde{c}_n .*

Proof. As in Lemma 4.6, we can choose geodesics $\tilde{g}_i : [0, r_i] \rightarrow \tilde{S}_n$ of length r_i passing though \tilde{q}_i , with $\tilde{g}_i(0) \in \tilde{c}_\epsilon$ and $\tilde{g}_i(r_i) \in \tilde{c}_n$ for $i = 1, 2$. Also, we have geodesic $\tilde{g}_o : [0, r_o] \rightarrow \tilde{S}_n$ with $\tilde{g}_o(0) = \tilde{g}_1(0)$ and $\tilde{g}_o(r_o) = \tilde{g}_2(r_2)$.

For $1 \leq i \leq 2$, there is r'_i such that $\tilde{q}_i = \tilde{g}_{s_i}(r'_i)$ with $0 < r'_i < r_i$. And since $l_i = \tilde{d}_n(\tilde{q}_i, \tilde{c}_\epsilon)$ and $r'_i = \tilde{d}_n(\tilde{g}_i(0), \tilde{p}_i)$, then $l_i \leq r'_i \leq l_i + r_3$.

First, assume that we have $r_o \leq r_1 \leq r_2$. In the isosceles triangle with three vertices $\tilde{g}_1(0), \tilde{g}_2(r_2)$ and $\tilde{g}_1(r_o)$, since $r'_1 = \tilde{d}_n(\tilde{q}_1, \tilde{g}_1(0)) = \tilde{d}_n(\tilde{g}_o(r'_1), \tilde{g}_1(0))$, by CAT(0) property of \tilde{S}_n , we have

$$\begin{aligned} \tilde{d}_n(\tilde{q}_1, \tilde{g}_o(r'_1)) &\leq \tilde{d}_n(\tilde{g}_1(r_o), \tilde{g}_2(r_2)) \\ &\leq \tilde{d}_n(\tilde{g}_1(r_o), \tilde{g}_1(r_1)) + \tilde{d}_n(\tilde{g}_1(r_1), \tilde{g}_2(r_2)) \leq (r_1 - r_o) + r_4 \end{aligned}$$

In the isosceles triangle with three vertices $\tilde{g}_2(r_2), \tilde{g}_1(0)$ and $\tilde{g}_2(r_2 - r_o)$, since $r'_2 - (r_2 - r_o) = \tilde{d}_n(\tilde{q}_2, \tilde{g}_2(r_2 - r_o)) = \tilde{d}_n(\tilde{g}_o(r'_2 - (r_2 - r_o)), \tilde{g}_1(0))$, by CAT(0) property of \tilde{S}_n , we have

$$\begin{aligned} \tilde{d}_n(\tilde{q}_2, \tilde{g}_o(r'_2 - (r_2 - r_o))) &\leq \tilde{d}_n(\tilde{g}_1(0), \tilde{g}_2(r_2 - r_o)) \\ &\leq \tilde{d}_n(\tilde{g}_1(0), \tilde{g}_2(0)) + \tilde{d}_n(\tilde{g}_2(0), \tilde{g}_2(r_2 - r_o)) \leq r_3 + (r_2 - r_o) \end{aligned}$$

Moreover, since r_3 and r_4 are \tilde{d}_n -lengths of \tilde{c}_ϵ and \tilde{c}_n , so we have

$$|r_1 - r_o| \leq r_3 + r_4 \text{ and } |r_2 - r_o| \leq r_3 + r_4$$

Consequently,

$$\begin{aligned}
\tilde{d}_n(\tilde{q}_1, \tilde{q}_2) &\leq \tilde{d}_n(\tilde{q}_1, \tilde{g}_o(r'_1)) + \tilde{d}_n(\tilde{g}_o(r'_1), \tilde{g}_o(r'_2 - (r_2 - r_o))) + \tilde{d}_n(\tilde{g}_o(r'_2 - (r_2 - r_o)), \tilde{q}_2) \\
&\leq ((r_1 - r_o) + r_4) + |(r'_2 - (r_2 - r_o)) - r'_1| + (r_3 + (r_2 - r_o)) \\
&\leq |r_1 - r_o| + |r'_2 - r'_1| + 2|r_2 - r_o| + r_3 + r_4 \\
&\leq 2(r_3 + r_4) + (2r_3 + |l_1 - l_2|) + 2 \cdot 2(r_3 + r_4) + r_3 + r_4 \\
&= |l_1 - l_2| + 9r_3 + 7r_4,
\end{aligned}$$

Second, for all other cases, similarly, we can always get:

$$\tilde{d}_n(\tilde{q}_1, \tilde{q}_2) \leq |l_1 - l_2| + 9r_3 + 7r_4.$$

□

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